# The Riemann zeta function on the critical line plotted with *Mathematica*

**Historical background**
In 1859 Riemann broke important new ground in several areas of mathematics by publishing the eight-page paper *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (On the Number of Primes Less than a Given Magnitude). In this paper he outlined a method to prove the prime number theorem conjectured by Gauss (1793) and Legendre (*c.* 1800). The prime number theorem states that $\pi(x)$, the number of prime numbers less than or equal to a number $x$, is roughly $x/\log x$ (in other words, the "probability" that a number $n$ is prime is about $1/\log n$). There were many gaps in Riemann's outline and little progress was made for about 30 years, but much effort by various mathematicians finally culminated in independent proofs by Hadamard and de la Vallée-Poussin in 1896.

Riemann's method is based on the function

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

which is called `Zeta[s]` in *Mathematica*. Euler had studied this function earlier and proved the remarkable identity

$$\sum_{k=1}^{\infty} \frac{1}{k^s} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}}$$

where the product runs over all prime numbers. Euler considered the $\zeta$ function only for real values of $s$. Riemann's great insight was to study this function for *complex* values of $s$ and to use the powerful methods of complex analysis, which led him to connect the complex zeros of this function and prime numbers:

$$\pi(x) \sim \mathrm{li}(x) - \sum_{\rho} \mathrm{li}(x^{\rho}) + O(\sqrt{x})$$

where the sum is over all of the complex zeros $\rho$ of the $\zeta$ function and $\mathrm{li}(x)$ is the logarithmic integral function, that is, the principal value of $\int_0^x 1/\log t\, dt$. An important issue here is the size of $\sum_\rho \mathrm{li}(x^\rho)$ and in particular the magnitudes of the $x^\rho$ or the real parts of the zeros. In fact it was subsequently shown that the prime number theorem follows from the fact that the $\zeta$ function has no complex zeros $\rho$ with $\mathrm{Re}(\rho) \geq 1$.

Figure 1 is a *Mathematica* plot of $\mathrm{li}(x) - \pi(x)$. One question is how $\mathrm{li}(x) - \pi(x)$ behaves for large $x$. From the plot one would suspect that this quantity is positive (Riemann was the first to conjecture this) and grows slowly with $x$. It turned out that at least the first of these suspicions was incorrect: Littlewood (1914) showed that the quantity must change sign infinitely often as $x$ increases. However, the first sign change has never been seen explicitly; all that is known is that it occurs for some value of $x$ below $10^{400}$.

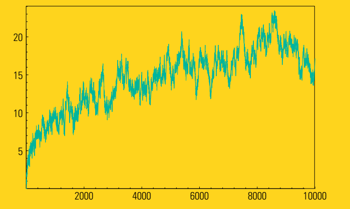

**Figure 1:** `Plot[LogIntegral[x]-PrimePi[x], {x,1,10000}]`

With regard to the magnitude of $\mathrm{li}(x) - \pi(x)$, it is strongly believed that this quantity is of order at most $\sqrt{x}\log x$. What is interesting is that the actual bound is related to the location of the zeros of the $\zeta$ function.

**The Riemann hypothesis**
Riemann's paper contains one of the most celebrated of all mathematical problems—the so-called Riemann hypothesis that *all the complex zeros of $\zeta(s)$ lie on the so-called "critical line"* $\mathrm{Re}(s) = 1/2$. If this is true, the prime numbers are in some sense distributed as evenly as possible and $\mathrm{li}(x) - \pi(x)$ grows no more rapidly than $O(\sqrt{x}\log x)$.

Little progress toward this result has been achieved since Riemann first conjectured it, although many first-class mathematicians have worked on it at some time in their careers. This is not to say that such work has been fruitless. On the contrary, such work has led to significant advances in many branches of mathematics, including complex analysis, Fourier analysis, and analytic number theory.

**Known results**
Some of the results about the zeros of the $\zeta$ function are almost trivial; others comprise some of the deepest theorems in mathematics and are the work of some of the most talented mathematicians of this century. For example, it is known that:

- $\zeta(s)$ has infinitely many real zeros, namely, at the negative even integers.
- Each of the complex zeros lies within the "critical strip" $0 < \mathrm{Re}(s) < 1$.
- For each complex number $\rho$ that is a zero of $\zeta$, the numbers $1 - \rho$, $\overline{\rho}$, and $1 - \overline{\rho}$ are also zeros.
- At least 40% of the zeros lie on the critical line and nearly all of the zeros that are not on it are arbitrarily close to it.
- The average distance between successive zeros in the vicinity of $\frac{1}{2} + it$ is about $2\pi/\log\frac{t}{2\pi}$.
- Each of the first 1,500,000,001 complex zeros lies on the critical line.

These and other results lead most mathematicians to believe the Riemann hypothesis to be true, but it is still unproven.

Figure 2 is a contour plot of the logarithm of the absolute value of the $\zeta$ function on the rectangle with vertices at $5, 5+40i, -7+40i$, and $-7$. Notice the pole at $s = 1$ and the zeros along the negative real axis as well as the zeros along the critical line.

**The Riemann-Siegel formula**
The most efficient way known to compute a single value of $\zeta(s)$ "far up" in the critical strip was published in 1932 by Siegel, who reconstructed it from Riemann's unpublished notes; the method is now known as the Riemann-Siegel formula. For $t$ real and positive, one defines the functions

$$\vartheta(t) = \arg\Gamma\left(\tfrac{1}{4} + \tfrac{it}{2}\right) - \frac{t\log\pi}{2}$$

and

$$Z(t) = \zeta\left(\tfrac{1}{2} + it\right)e^{i\vartheta(t)}$$

and extends them by analytic continuation to the complex plane cut along the imaginary axis from $\frac{i}{2}$ to $i\infty$ and from $-\frac{i}{2}$ to $-i\infty$. The Riemann-Siegel formula is an asymptotic formula for $Z(t)$, the main term of which is

$$\sum_{1 \leq n < \sqrt{t/2\pi}} 2\cos[\vartheta(t) - t\log n]/\sqrt{n}$$

The functions $Z(t)$ and $\vartheta(t)$ are called `RiemannSiegelZ[t]` and `RiemannSiegelTheta[t]` in *Mathematica*.

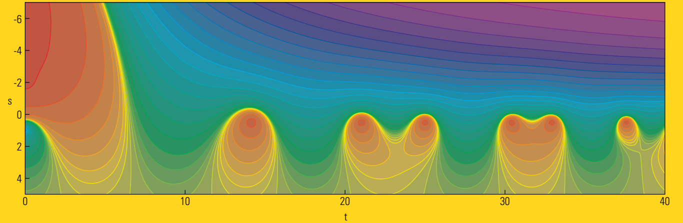

**Figure 2:** `ContourPlot[Log[Abs[Zeta[s + I t]]],{s,-7,3},{t,0,40}]`

Note that there is a trivial relationship between the complex zeros of $\zeta$ and the real zeros of $Z$ and that for $t$ real

$$\left|\zeta\left(\tfrac{1}{2} + it\right)\right| = |Z(t)|$$

The splitting of $\zeta(s)$ into the two functions $\vartheta(t)$ and $Z(t)$ is convenient for several reasons. First, for real values of $t$ (*i.e.*, along the critical line) $\vartheta(t)$ and $Z(t)$ are real; therefore, complex arithmetic can be completely avoided and, by the intermediate value theorem, the zeros of $Z(t)$ can readily be found by locating the sign changes of $Z(t)$. Second, there is an asymptotic formula for $\vartheta(t)$ such that, for any given precision, as $t$ gets larger, $\vartheta(t)$ actually takes *less* time to evaluate and, using the Riemann-Siegel formula, the time to compute $Z(t)$ only increases as $\sqrt{t}$. With previously known methods the time increases as $t$. Thus one can extend computation of $\zeta(1/2+it)$ to *much* larger values of $t$. Finally, to verify the Riemann hypothesis in the range $0 < t < T$, it is necessary to know that the number of zeros on the critical line is the same as the number in the critical strip. Counting sign changes of $Z(t)$ provides the one number and

$$\frac{\mathrm{Re}[\vartheta(T) - i\log\zeta(1/2 + iT)]}{\pi} + 1$$

provides the other.

In 1903 Gram observed a simple rule for finding the sign changes in $Z(t)$: *The zeros of $Z(t)$ appear to alternate with the zeros of* $\sin\vartheta(t)$. Knowing the Riemann-Siegel formula, we can give a heuristic argument for why Gram's "law", as it is called, might hold. If we consider only the $n = 1$ term of the sum in the Riemann-Siegel formula, we get $Z(t) \approx 2\cos\vartheta(t)$ that has its extrema at the zeros of $\sin\vartheta(t)$. It is known that there are places where Gram's law fails, but it is amazing how rarely these failures actually occur. Just look at the behavior of the plot of $Z(t)$ below and see how crude an approximation $2\cos\vartheta(t)$ is to it; $Z(t)$ takes on values that are well outside the range $-2$ to $2$.

**The Lindelöf hypothesis**
Another hypothesis associated with the $\zeta$ function is the so-called Lindelöf hypothesis. Notice that the graph of $Z(t)$ has places where it gets much farther from the axis than it normally is. The Lindelöf hypothesis says that the size of these "glitches" grows more slowly than any positive power of $t$ as $t$ goes to $\infty$ along the real axis. It is known that they grow more slowly than $t^{0.16}$, but it is also known that they are not bounded. From the plot here it indeed appears that $|Z(t)|$ is not bounded, but it is not clear how slowly it grows. Although the Lindelöf hypothesis is weaker than the Riemann hypothesis, it too has certain implications regarding the distribution of the prime numbers.

The Hadamard product formula (1893) essentially expresses the $\zeta$ function as an infinite-degree polynomial in factored form. From this it can be seen that the large glitches with which the Lindelöf hypothesis is concerned are related to the distances between nearby zeros. Indeed in the plot of $Z(t)$ below it appears that, at least in a local sense, the largest values of the function occur between zeros that are widely separated. Odlyzko (1987) found that the distribution of the spacings between successive zeros appears to be close to a certain distribution which governs the spacing between eigenvalues of some random matrices. This result suggests that the zeros of the $\zeta$ function might correspond to the eigenvalues of a Hermitian operator. If this could indeed be shown, then the Riemann hypothesis would be established.

**About the plot**
The plot below is of `RiemannSiegelZ[t]` along five sections of the positive real axis using the same horizontal and vertical scale. Each section of the plot can be generated with the command `Plot[RiemannSiegelZ[t], {t, tmin, tmax}]`, with appropriate values for $tmin$ and $tmax$. The curve is colored cyclically from green to blue using the values of `RiemannSiegelTheta[t]`.

Noteworthy features of the plot include:

- As $t$ gets larger the *average* spacing between successive zeros gets closer to zero (this is a theorem of Littlewood).
- The spacing between successive zeros varies a great deal. In fact the spacing between some zeros is known to be as small as $0.00031$ times the average spacing.
- With the exception of the first local maximum, each local maximum occurs above the axis and each local minimum occurs below the axis. The Riemann hypothesis implies that, aside from the exception noted, this is always true.
- There are places where the graph *almost* turns around before it crosses the axis. These places can be thought of as near counterexamples to the Riemann hypothesis. Such behavior is known as "Lehmer's phenomenon".

**This poster is dedicated to the memory of Jerry B. Keiper (1953–1995), leader of the numerics research and development group at Wolfram Research and author of many original numerical algorithms in *Mathematica*. A first version of this poster was created in 1990 by Jerry Keiper and others, making use of Keiper's extensive work on the Riemann zeta function.**

**Jerry Keiper's obituary appears on the World Wide Web in http://www.wri.com/keiper/obituary.html**

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Only known negative local maximum {2.47572..., -0.52625...}

First zero (14.13472...)

**1859**
Riemann (3 zeros)

**1903**
Gram (10 zeros)

**1918**
Backlund (79 zeros)

**1925**
Hutchinson (138 zeros)

**1936**
Titchmarsh and Comrie (1,041 zeros); first time electronic machines used on this problem

**1953**
Turing (1,104 zeros); first time an electronic computer programmed to work on this problem

Largest negative value so far {6925.62193..., -15.95467...}

Near counterexample to the Riemann hypothesis {7005.08185..., 0.00396...}

**1956**
D. H. Lehmer (25,000 zeros)

**1958**
Meller (35,337 zeros)

**1966**
Lehman (250,000)

**1969**
Rosser, Yohe, and Schoenfeld (3,500,000 zeros)

**1979–1986**
Brent, van de Lune, te Riele, and Winter (1,500,000,001 zeros)

**1994–1995**
Odlyzko and Schönhage ($1.8\times10^8$ zeros around the $10^{20\text{th}}$ zero which is at 15202440115920747268.6290299...)